Linear Ordinary Differential Equations and Fermat Equations

線形常微分方程式とフェルマ方程式

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A theorem analogous to Picard's theorem on representation of a plane algebraic curve of genus greater than 1 with meromorphic functions will be proved. Its enunciation will be done for elements in a Fuchsian extension defined in this note instead of considering for meromorphic functions. As seen straightforwardly, a differential extension generated with solutions of linear ordinary differential equations turns out to be Fuchsian, hence the theorem deduces a corollary that solutions of linear ordinary differential equations substaintially satisfy no Fermat equations.

種数＞１の平面代数曲線の有理型関数による表現に関するピカールの定理の類似が証明される。命題は有理型関数に対してではなく、このノートで定義されるフックス拡大の要素に対して記述される。線形常微分方程式の解で生成される微分拡大はフックス拡大であり、この定理は系として、線形常微分方程式の解はフェルマ方程式を本質的に満足しないという結果を導く。

Keywords: Linear Ordinary Differential Equation, Fuchsian Extension, Prime Divisor, Picard's Theorem, Picard-Vessiot Extension, Strongly Normal Extension

1 Fuchsian extensions

Picard's theorem [P] states: Suppose two meromorphic functions u, v sataisfy the irreducible algebraic relation of degree m

$$F(u, v)=0.$$

Then the genus of the algebraic relation must be 0 or 1. Single-valuedness has been a little investigated in the theory of differential algebra, for instance [M] and [B]. This paper attempts to propose a concept relating to algebraic differential equations with no movable algebraic branches and apply it to show an analogy of Picard's theorem.

Let K be a differential field of characteristic 0 with differentiation D and let C_R denote the constant field of a differential extension R/K. In the following we set $C=C_K$ and assume it to be algebraically closed. A prime divisor of a finitely generated field extenstion R/K in this paper means an equivalence class of discrete valuation of rank 1 of R/K.

A differential extension R/K which is finitely generated as a field extension will be said to be *Fuchsian* if there is a set Π of prime divisors of R/K such that each valuation ring O_P of $P \in \Pi$ is stable under the differentiation D and every element of $R\backslash\overline{K}$ has a polar prime divisor in Π.

Here we will discuss some basic properties of Fuchsian extensions.

If R/K is Fuchsian and S is an intermediate differential field between R and K, then S/K is Fuchsian. In fact let Π be the required set of prime divisors of R/K in the definition. Then clearly $D(O_P \cap S) \subset O_P \cap S$ and if $P \in \Pi$ is a polar prime divisor of $u \in S\backslash\overline{K}$ then so is the restriction to S of P, $P|_S$. Hence we may take the set of prime divisors of S/K

$$\{P|_S \mid \text{some element of } S \text{ has a pole at } P \in \Pi\}$$

as a required one.

A differential algebraic function field of one dimension R/K which is Fuchsian has no movable singularities in the sense of Matsuda [M], namely being a differential extension of which every valuation ring is stable under D. In fact let P be a prime divisor of R/K. There is an element $u \in R\backslash\overline{K}$ with its pole only at P. By definition we see $P \in \Pi$, hence $DO_P \subset O_P$, which indicates the assertion. The converse can be readily seen.

A differential extension R/K is said to depend rationally on arbitrary constants if there exists a differential extension L/K such that R and L are free over K and $LR=LC_{LR}$ holds [N]. If this is the case R/K is Fuchsian with a set of prime divisors Π, which is defined as follows. Let $c_1, c_2, \ldots, c_n$ be a transcendence base of C_{LR} over C_L and Π define the set consisting of restrictions to R of prime divisors of LR/L_i $(1\leq i\leq n)$, where $L_i = L(c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n)$. Noting that LR/L_i has no movable singularities in Matsuda's sense, for $u \in R\backslash\overline{K} \subset LR\backslash\overline{L}$ for some i there is a prime divisor P_i of LR/L_i where u has a pole. The restriction $P_i|_R$ is our prime divisor.

Any strongly normal extension R/K is Fuchsian since it depends rationally on arbitrary constants viewing the interpretation of [BB]: R/K is by definition strogly normal if the quotient field of $R \otimes_K R$ is generated with constants over $1\otimes R$ provided K being algebraically closed in R.

2 Theorem

In the sequel we assume that K is algebraically closed in R and $C=C_K$. Let R/K be a differential field extension. Then its differential module $\Omega_{R/K}$ has a C_R-linear operator D^1 (the Lie derivative) characterized by

$$D^1(adb) = D(a)db + adDb \quad (a, b \in R),$$

which in particular satisfies $D^1d=dD^1$ on R. According to [R], $D^1(adb) = d(aDb)$ holds provided $a, b \in R$ are algebraically dependent over C.

The following is an analogue of the Picard's theorem on analytic representation of a plane curve.

Theorem Let R/K be a Fuchsain differential field extension with Π, a set of prime divisors of R/K. Suppose that there exists a subfield S of R which is a one-dimensional algebraic function field over C with genus greater than one. Then $C_R \neq C$.

Proof Let $udv \in \Omega_{S/C} \subset \Omega_{KS/K}$ be regular. We then have $D^1(udv) = d(uDv)$. Assume that uDv is transcendental over K. Then there exists a prime divisor $P \in \Pi$ at which uDv has a pole. Let $t \in KS$ be a prime element associated with the restriction of P to KS. By assumption there is an element $w \in O_P$ with $udv = wdt$, whence $D^1(udv) = D(w)dt + wdDt$. Since $D_t, D_W \in O_P$, it follows $d(uDv) = D^1(udv) \in O_{Pd}O_P$ and a contradiction. Thus $uDv \in K$. Since S and K are linearly disjoint over C and $KS \subset R$, KS/K has the same genus as S/C and is Fuchsian, therefore having no movable singularities in the sense of Matsuda as mentioned in the introduction. By the theorem of Poincaré in [N] or [M], $C_{KS} \neq C$.

Corollary Let n be an integer greater than 3. Suppose that R/K has no movable algebraic singularities with $C_R = C$ and there are elements u, v in R with $u^n + v^n = 1$. Then they in fact are algebraic over K.

Proof Assume that u is transcendental over K, and let $S = C(u, v)$. Then S is a one-dimensional algebraic function field over C with genus greater than 1. By the theorem we have $C_R \neq C$, a contradiction.

Corollary Let n be an integer greater than 2. Suppose that R/K is a Picard-Vessiot extension and there are elements u, v in R with $u^n + v^n = 1$. Then they in fact are algebraic over K.

Proof R/K is generated by a fundamental system of solutions, Φ, of the system of linear differential equations $D\Phi = A\Phi$ over K. Since the quotient field of $R \otimes_K R$ is generated with constants, R/K has nomovable algebraic singularities. So, it remains to prove in the case of $n=3$. As seen above $S = K(u, v)/K$ has no movable singularities in the sense of Matusda with genus 1. Hence it is an elliptic function field over K ([N] or [M]), whence an abelian extension. Noting the following Remark, S must agree with K, which shows our assertion.

Remark Suppose R/K is a Picard-Vessiot extension with K being algebraically closed in R and S/K an abelian extension with $C_{RS} = C$. Then R and S are linearly disjoint over K. In fact, the differential Galois group $G(R \cap S/K)$ is the C-homomorphic image of $G(R/K)$ as well as $G(S/K)$ (Theorem 4 in [K, p.401]). Hence it is affine as well as complete, consequently $R \cap S = K$ by [K, pp.359 and 377]. Since by Theorem 5 in [K, p.403] $G(RS/S) \approx G(R/K)$ it follows trans.deg RS/S = trans.deg R/K, which implies R and S are free over K, hence they are linearly disjoint over K since R/K is regular.

We shall end this note by explaining our framework applies to another proof of Sperber's theorem [S]: Let each of nonzero $y_1, y_2, \ldots, y_n (n>1)$ satisfy some linear ordinary differential equation over K, and suppose they fulfill $y_1 = y_2^{m_2} \cdots y_n^{m_n}$, where the m_i are positive integers. Assume N, the order of the linear ordinary differential equation over K satisfied by y_1, does not exceed $\min\{m_2, \ldots, m_n\}$ then all the Dy_i/y_i are algebraic over K. In fact suppose the converse, namely, some Dy_j/y_j is transcendental over K. Since the differential extension $R = K\langle y_1, y_2, \ldots, y_n\rangle/K$ is Fuchsian, there is a polar prime divisor P of Dy_j/y_j with the valuation ring O_P being stable under D. Let v and t be the valuation and a prime element for P. If $v(Dt)>0$ then $v(Dy_j/y_j) \geq 0$ because describing $y_j = t^r z (v(z)=0)$ we have $v(Dy_j/y_j) = v(rDt/t + Dz/z) \geq 0$. This is absurd. Hence $v(Dt)=0$.

This time $v(Dz)=v(z)-1$ holds for $z \in R$ with $v(z) \neq 0$. Since inequality $v(y_i)<0$ would derive that y_i satisfies no linear ordinary differential equation over K, it follows $v(y_i) \geq 0$ for any i and clearly $v(y_j)>0$. Then

$$N>v(y_1)=m_2v(y_2)+\cdots+m_nv(y_n) \geq m_jv(y_j) \geq m_j,$$

and so that $N>\min\{m_2,\cdots, m_n\}$, a contradiction, which completes the proof.

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