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On Team Decision Theory

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Abstract

This short note tries to revive team decision theory by Ho&Chu(1972), original contributions going back to J.Marschak(1955) and R.Radner(1962), for decision making problems in microeconomic formulation. Main characteristics of team decision is, due to the basis on statistical decision theory, that optimal decision has random part inside together with deterministic part.

1. Introduction

This short note attempts to revive team decision theory of Ho&Chu type(1972), original contributions going back to J.Marschak(1955) and R.Radner(1962), for further study of microeconomic application in related fields. Implication of team decision is, due to the basis on statistical decision theory, that optimal decision has random part inside together with deterministic part. The present author once tried to apply the theory to the macro-economic stabilization policy by the use of econometric models.(See H.Kosaka(1987))

2. Team Decision Theory of Ho&Chu: brief review and extensions

Ho & Chu Team Decision Theory

We briefly review the Ho & Chu type team decision theory and make extensions. We denote by "e" the external environment for the team and assume that is obeys:

$$e \sim N(0,\Sigma) \qquad \Sigma > 0 \tag{2.1}$$

The i-th member of the team would act based on information $I_i(t)$, which has been observed by the member and has obtained from the communication with the other members. Hence the information $I_i(t)$ would depend not only on the external environment, but also on the other members' actions. Hence we write $I_i(t)$ in the following.

$$I_i = H_i e + \sum_{j=1}^{M} D_{ij} z_j \qquad \text{j=1,2,...,M} \tag{2.2}$$

M : number of the team member

z_j : j-th member's control behavior

H_i : $h_i \times n$ order matrix($h_i < n$) of full rank(i.e rank $H_i = h_i$)

Now we restate some basic definition for the team decision problem.[Y.C.Ho & K.C.Chu 1972]

Definition 1: If $D_{ij} \neq 0$, the j-th member is said to relate to the i-th member.(jRi)

Definition 2: If jRi or if there exist r,s,t,..,k such that jRr, rRs, sRt,.., kRi, the j-th member is said to be a precedent of the i-th member.

The i-th member is supposed to determine his control action based on his information:

$$z_i = r_i(I_i) \qquad i=1,2,...,M \qquad (2.3)$$

in which we call r_i a control law or decision rule. In deciding his control law r_i, each member cooperates with each other to minimize the common objective function in the following way.

[Team Decision Problem]

$$\min_{z_i} E(F|I_i)\text{ [1]} \qquad i=1,2,...,M \qquad (2.4)$$

$$F = z'Qz + z'Se + z'C \qquad Q>0$$

$$z = \begin{pmatrix} z_1 \\ . \\ z_i \\ . \\ z_M \end{pmatrix} = \begin{pmatrix} r_1(I_1) \\ . \\ r_i(I_i) \\ . \\ r_M(I_M) \end{pmatrix}$$

where $E(F|I_i)$ denotes conditional expectation of F given I_i, and F is quadratic loss of z. Prior to solve the problem (2.4), we distinguish static team with dynamic one.

Definition 3: If $I_i = H_i e$ (i=1,2,..,M), the team is said as static team, and otherwise as dynamic team.

Static team ensures to have the unique solution [Theorem, Y.C.Ho & K.C.Chu 1972], and the following condition give us the unique solution for the problem (2.4).[2]

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[1] In determining the control law r_i from the statistical decision theory, so-called "Extensive Form" is used.

[2] In multivariate normal distribution $x \sim N(\mu, \Sigma)$, the conditional expectation of x_1 given x_2 states $E(x_1|x_2) = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$.

$$Q_{ii}r_i(I_i)+\sum_{j\neq i}^{M}Q_{ij}E[r_j(I_j)|I_i]+S_iE[e|I_i]+C_i=0 \qquad (2.5)$$

$$Q=(Q_{ij}) \qquad S=\begin{pmatrix}S_1\\S_2\\.\\S_M\end{pmatrix} \qquad C=\begin{pmatrix}C_1\\C_2\\.\\C_M\end{pmatrix}$$

The theorem states that the condition (2.5) with the following linear control law gives us the optimal solution.

$$z_i=A_iI_i+b_i \qquad i=1,2,..,M \qquad (2.6)$$

Then the equation (2.5) becomes:

$$\begin{aligned}&Q_{ii}(A_iI_i+b_i)+\sum_{j\neq i}^{M}Q_{ij}E[A_jI_j+b_j|I_i]+S_iE[e|I_i]+C_i\\&=Q_{ii}(A_iI_i+b_i)+\sum_{j\neq i}^{M}Q_{ij}A_jE[I_j|I_i]+\sum_{j\neq i}^{M}Q_{ij}b_j+S_iE[e|I_i]+C_i\\&=Q_{ii}(A_iH_ie+b_i)+\sum_{j\neq i}^{M}Q_{ij}A_jE[I_j|I_i]+\sum_{j\neq i}^{M}Q_{ij}b_j+S_iE[e|I_i]+C_i=0\end{aligned} \qquad (2.7)$$

[3]

Substituting (2.6) into (2.7), we have the following two set of linear simultaneous equations for the coefficients of linear control law.

$$\sum_{j=1}^{M}Q_{ij}A_j(H_j\Sigma H_i^{'})=-S_i\Sigma H_i^{'} \qquad i=1,2,..,M \qquad (2.8)$$

$$\sum_{j=1}^{M}Q_{ij}b_j=-C_i \qquad i=1,2,..,M \qquad (2.9)$$

When we look at the equation (2.9), the coefficients b_j ($i=1,2,..,M$) are said to be determined by direct deterministic minimization of objective function F with respect to z. Therefore the solution with the linear control law is composed of (i) b_j and (ii) stochastic part, which expresses learning process during the control period depending on stochastic terms.

An Extension to Non-zero External Environment

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[3] As we have two kinds of multivariate distributions $\begin{pmatrix}H_je\\H_ie\end{pmatrix}\sim N\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}H_j\Sigma H_j^{'} & H_j\Sigma H_i^{'}\\H_i\Sigma H_j^{'} & H_i\Sigma H_i^{'}\end{pmatrix}\right\}$ and $\begin{pmatrix}e\\I_i\end{pmatrix}=\begin{pmatrix}e\\H_ie\end{pmatrix}\sim N\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\Sigma & \Sigma H_i^{'}\\H_i\Sigma & H_i\Sigma H_i^{'}\end{pmatrix}\right\}$; then we have $E(I_j|I_i)=H_j\Sigma H_i^{'}(H_i\Sigma H_i^{'})^{-1}H_ie$ and $E[e|I_i]=\Sigma H_i^{'}(H_i\Sigma H_i^{'})^{-1}H_ie$ by footnote 2.

For more general case of the external environment obeying multivariate normal distribution with nonzero mean in place of (2.1), we have the following corollary.

Corollary. The static team with

external environment: $e \sim N(\mu,\Sigma) \quad \mu \neq 0$ (2.10)

information structure: $I_i = H_i e \quad i = 1,2,..,M$, (2.11)

has the linear control law:

$$r_i = A_i I_i + b_i \tag{2.12}$$

with coefficients (A_i, b_i) having the following equations.

$$\sum_{j=1}^{M} Q_{ij} A_j (H_j \Sigma H_i^{'}) = -S_i \Sigma H_i^{'} \quad i = 1,2,..,M \tag{2.13}$$

$$\sum_{j=1}^{M} Q_{ij} b_j = -C_i - (\sum_{j=1}^{M} Q_{ij} A_j H_j + S_i)(\mu - \Sigma H_i^{'} (H_i \Sigma H_i^{'})^{-1} H_i \mu) \quad i = 1,2,..,M \tag{2.14}$$

Proof. If we put $e^* = e - \mu$, existence of unique solution is assured as in the same argument of Ho & Chu. [Lemma,Y.C.Ho & K.C.Chu 1972] And derivation of (2.13)-(2.14) is obvious from conditional expectation of multivariate statistics. Q.E.D.

To obtain numerical solution from system of equation (2.8)-(2.9) or (2.13)-(2.14), we transform it into linear equation with respect to unknown variables. Equation (2.8) or (2.13) can be rewritten down in a more compact form.

$$\sum_{j=1}^{M} Q_{ij} A_j E_{ji} = F_i \qquad i = 1,2,..,M \tag{2.15}$$

$$E_{ji} = H_j \Sigma H_i^{'} \qquad F_i = -S_i X H_i^{'}$$

Using Kronecker product, (2.15) is transformed into the following,

$$\begin{pmatrix} Q_{11} \otimes E_{11} & Q_{12} \otimes E_{12} & . & Q_{1M} \otimes E_{1M} \\ Q_{21} \otimes E_{21} & Q_{22} \otimes E_{22} & . & Q_{2M} \otimes E_{2M} \\ . & . & . & . \\ Q_{M1} \otimes E_{M1} & Q_{M2} \otimes E_{M2} & . & Q_{MM} \otimes E_{MM} \end{pmatrix} \begin{pmatrix} A_1^* \\ A_2^* \\ . \\ A_M^* \end{pmatrix} = \begin{pmatrix} F_1^* \\ F_2^* \\ . \\ F_M^* \end{pmatrix} \tag{2.16}$$

where A_i^* and F_i^* are column vector obtained from the row of A_i and F_i

respectively.[P.Lancaster 1969]

Partially Nested Information Structure

To broaden applicability of static team, Y.C.Ho and K.C.Chu worked out device for a general dynamic team with information (2.2), by reducing dynamic team problem to a static team problem by means of the concept of "partially nestedness" of information structure of the team.[Y.C.Ho & K.C.Chu 1972] The concept is stated below.

Definition 4: An information structure $[I_i : i = 1,2,..M]$ is called partially nested if the precedence of j to i implies $I_j = P_{ji} I_i$.

Then, they have proved, if an information structure of dynamic team is partially nested, the optimal control law of the dynamic team can be worked out from that of the corresponding static team. [Theorem 2, Y.C.Ho & K.C.Chu 1972] Then, K.C.Chu has shown the way of constructing dynamic team with partially nested information structure of the general dynamic team which have no partially nested information structure.[K.C.Chu 1972]

Deducing Equivalent Control

And he indicated a possibility of obtaining the optimal control law of the general dynamic team, by the use of concept of "equivalent control" which is stated below, from the corresponding dynamic team with partially nested information structure.[Theorem 2, K.C.Chu 1972]

In what follows we argue a method of deducing unknown control law of the dynamic team based on the concept of equivalency control. For this purpose we first give the definition of "equivalency." Prior to give the definition, we remark that the control z_i of dynamic team can be expressed by external environment "e" with arranging team's members order by their causal relation;

$$z_i = p_i(e) = p_i^{(1)} e + p_i^{(2)} \qquad i = 1,2,..,M . \tag{2.17}$$

The definition of "equivalency" is the following.

Definition 5: If $p_i(e) = \tilde{p}_i(e)$ for all i, $\tilde{p}_i$ is equivalent to p_i.

Suppose the equivalent control $\tilde{p}_i$ is known in advance on the basis that the corresponding team is dynamic team with partially nested information structure. Then the information I_i of the unknown control becomes known by substituting the relation $p_i(e) = \tilde{p}_i(e)$:

$$I_i = H_i e + \sum_{j=1}^{M} D_{ij} p_j(e) = H_i e + \sum_{j=1}^{M} D_{ij} \tilde{p}_i(e) \\ = g_i(1)e + g_i(2) = g_i(e) \tag{2.18}$$

Then we have the following theorem concerning the optimal solution of the general dynamic team.

Theorem 1. Under the known equivalent control $\tilde{p}_i(e)$, the linear control law $z_i = A_i I_i + b_i$ with

$$A_i = p_i(1) p_i^+(1) p_i(1) g_i^+(1) \tag{2.19}$$

$$b_i = p_i(2) - p_i(1) p_i^+(1) p_i(1) g_i^+(1) g_i(2) \tag{2.20}$$

in which $p_i^+(1)$ and $g_i^+(1)$ are Penrose inverse matrix of $p_i(1)$ and $g_i(1)$ respectively, is the optimal solution of dynamic team decision problem (2.4).

Proof. The sufficiency condition in K.C.Chu ensures to give us the optimal solution [Theorem 2, K.C.Chu 1972]. In order to check the condition, we are going to put the unknown control law r_i linear with respect to I_i:

$$z_i = r_i(I_i) = A_i I_i + b_i \qquad i = 1,2,..,M\,. \tag{2.21}$$

Since we know the information from the known equivalent control as in (2.18), we have:

$$\tilde{p}_i(e) = \tilde{p}_i(1)e + \tilde{p}_i(2) = p_i(e) \\ = A_i I_i + b_i = A_i(g_i(1)e + g_i(2)) + b_i = A_i g_i(1)e + (A_i g_i(2) + b_i) \tag{2.22}$$

For satisfying the relation (2.22), we take Penrose inverse matrix for the coefficient (A_i, b_i). Generally the solution of matrix equation $AZB = C$ is expressed by $Z = A^+ C B^+$ where A^+ and B^+ are Penrose inverse matrix of A and B respectively. [Section 3.4.8,C.R.Rao and S.K.Mitra 1971] Using the fact, the solution of matrix equation $\tilde{p}_i(1) Z g_i(1) = \tilde{p}_i(1)$ is $Z = \tilde{p}_i^+(1)\tilde{p}_i(1) g_i^+(1)$. Hence, letting (A_i, b_i) be;

$$A_i = \tilde{p}_i(1)\tilde{p}_i^+(1)\tilde{p}_i(1) g_i^+(1) \tag{2.23}$$

$$b_i = p_i(2) - \tilde{p}_i(1)\tilde{p}_i^+(1)\tilde{p}_i(1)g_i^+(1)g_i(2)\,, \tag{2.24}$$

the relation (2.22) is satisfied. Thus the reducibility condition is assured. Q.E.D.

Thus we mentioned general method of obtaining the control law of dynamic team based on the known equivalent control. Yet, if $g_i(1)(i=1,2,..,M)$ are square and non-singular, the coefficient are determine simply:

$$A_i = \tilde{p}_i(1)g_i^{-1}(1) \tag{2.25}$$

$$b_i = \tilde{p}_i(2) - \tilde{p}_i(1)g_i^{-1}(1)g_i(2) \tag{2.26}$$

The squareness requires that $\dim I_i = \dim e$ for all i, which means every member uses all the information on the external environment. Hence this is a special case.[See Corollary, K.C.Chu 1972]

3. Conclusions

This short note argues an analytical framework, based on the team decision theory, of incorporating an appropriate information of an economic model into determination of economic variables, in which we took into consideration difference of information utilized in economic agents. The empirical application will be postponed in near future.

Reference

1)Chu,K.C.,1972,"Team Decision Theory and Information Structures in Optimal Control Problems-Part II," IEEE Transactions on Automatic Control,Vol.AC-17,pp22-28.

2)DeGroot,M.H.,1970,Optimal Statistical Decision Theory, MacGraw-Hill,New York.

3)Ho,Y.C. and K.C.Chu,1972,"Team Decision Theory and Information Structures in Optimal Control Problems-Part I," IEEE Transactions on Automatic Control,Vol.AC-17,pp15-22.

4)Marschak,J.,1955,"Elements for a Theory of Teams," Management Science,Vol.1,pp.127-137.

5)Kosaka,H.,1987,"Making Effective Use of Information in Macroeconomic Policy Formation", The Proceedings of 5-th IFAC/IFORS Conference on

Dynamic Modelling and Control of National Economies, Pergamon Press.
6)Lancaster,P.,1969,Theory of Matrices,Academic Press,Inc.,New York.
7)Rao,C.R. and S.K.Mitra,1971,Generalized Inverse of Matrices and its Applications, John Wiley and Sons, Inc., New York.
8)Radner,R.,1962,"Team Decision Problems," Annals of Mathematical Statistics,Vol.33,no.3,pp857-881.
9)Tinbergen,J.,1954,Centralization and Decentralization in Economic Policy, North-Holland Publishing Company, Amsterdam.